

Study of Effect in Convergence of a Solution with the Variation in Accelerating Factor in Successive Over Relaxation Method

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ABSTRACT: In using Successive over Relaxation method to obtain approximate solution of elliptic or parabolic type partial differential equation, the choice of accelerating factor plays an important role. It effects on convergence of the solution. This paper deals with the variation in the convergence of the solution according to changes in accelerating factor.

KEYWORDS: Accelerating Factor, Convergence, Heat equation, Successive over Relaxation

I. INTRODUCTION

When Successive over Relaxation method is used to obtain solution of elliptic or parabolic type partial differential equation, the efficiency of the method depends on the accelerating factor used. B.A.Carre [1] discussed how to obtain optimum value of accelerating factor for SOR in solving elliptic partial differential equations. A.K.Rigler [2] discussed the use of Kohn-Kato formula for an upper bound of an eigen values to obtain estimation of spectral radius for SOR. partial differential equation. Zhong- ZhiBai, Gene H.Golub and Michael K.Ng [3] present SOR accelerating scheme for the normal skew Hermitian splitting iteration method. H. E. Kulrsud [4] discussed the method of obtaining a good estimate of the optimum relaxation factor. A. Nicholls, B Honig [5] discussed an efficient algorithm for the numerical solution of the Poisson–Boltzmann equation by the method of successive over-relaxation. Shiming Yang,Matthias K.Gobbert [6] derived the optimal relaxation parameter to any space dimension and in three dimension by means of test calculations. Louis W Ehrlich [7] derived the method to determine a sequence of successive over relaxation factors D.M.Young and H.E.Eidson [8] discussed the optimum relaxation factor for the SOR method when the eigen values are complex. Hadjidimos [9] proposed the various algorithms associated explicitly or implicitly with the iterative methods. O.Axelsson [10] discussed an efficient preconditioning techniques for solving large sparse systems arised by discretization of boundary value problems for partial differential equations. S.Galanis,A.Hadjidimos,D.Noutsos [11] discussed how to obtain minimum value of spectral radius of the iterative matrix of a p-cyclic SOR method for the solution of linear system. M. Eicrmnnn. W. Neitharmmer.A.Ruttan [12] determine the optimal relaxation parameter of the p-cyclic SOR method assuming that the eigen values of J_p^p are contained in a real interval.

II. METHOD OF SUCCESSIVE OVER RELAXATION

Consider the heat equation:

$$c \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \text{ c being constant} \quad (1)$$

Let us divide (x,t) plane into smaller rectangle by means of the sets of lines.

$$x_i = x_0 + ih, \quad i = 0,1,2,\dots$$

$$t_j = jk, \quad j = 0,1,2, \dots$$

Using Finite difference analogue and Crank-Nicolson method [13] for (1)

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$$c \left(\frac{u_{i,j+1}^{j+1} - u_{i,j}^j}{k} \right) = \frac{1}{2} \left(\frac{u_{i+1,j+1}^{(j+1)} - 2u_{i,j+1}^{(j+1)} + u_{i-1,j+1}^{(j+1)}}{h^2} + \frac{u_{i+1,j}^j - 2u_{i,j}^j + u_{i-1,j}^j}{h^2} \right) \quad (2)$$

$$-\lambda u_{i-1,j+1} + (2 + 2\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + (2 - 2\lambda)u_{i,j} + \lambda u_{i+1,j} \quad (3)$$

$$\text{Where } \lambda = \frac{k}{ch^2}$$

Equation (3) is an implicit scheme, is called Crank-Nicolson formula and is convergent for all finite values of λ .

If there are N internal mesh points on each row, then formula (3) gives N simultaneous equations for the N unknowns in terms of the given boundary values. Similarly, the internal mesh points on all rows can be calculated.

From equation (3), we have

$$(1 + \lambda)u_{i,j+1} = u_{i,j} + \frac{\lambda}{2} [u_{i-1,j+1} + u_{i+1,j+1} + u_{i+1,j} + u_{i-1,j} - 2u_{i,j}] \quad (4)$$

$$\text{Let } c_i = u_{i,j} + \frac{\lambda}{2} [u_{i+1,j} + u_{i-1,j} - 2u_{i,j}] \quad (5)$$

From (3), we have

$$u_i = \frac{c_i}{(1+\lambda)} + \frac{\lambda}{2(1+\lambda)} [u_{i-1} + u_{i+1}] \quad (6)$$

$$u_i^{(n+1)} = \frac{c_i}{(1+\lambda)} + \frac{\lambda}{2(1+\lambda)} [u_{i-1}^n + u_{i+1}^n] \quad (7)$$

$$u_i^{(n+1)} = \frac{c_i}{(1+\lambda)} + \frac{\lambda}{2(1+\lambda)} [u_{i-1}^{(n+1)} + u_{i+1}^n] \quad (8)$$

Equation (7) is known as Gauss- Jacobi formula and (8) is known as Gauss- Seidal formula.

It can be shown that the scheme (8) converges for all finite values of λ and that it converges twice as fast as Jacobi scheme.

Rewriting equation (8) as

$$u_i^{(n+1)} = u_i^{(n)} + \left\{ \frac{c_i}{(1+\lambda)} + \frac{\lambda}{2(1+\lambda)} [u_{i-1}^{(n+1)} + u_{i+1}^n] - u_i^{(n)} \right\} \quad (9)$$

From (8),

$$u_i^{(n+1)} = u_i^{(n)} + \omega \left\{ \frac{c_i}{(1+\lambda)} + \frac{\lambda}{2(1+\lambda)} [u_{i-1}^{(n+1)} + u_{i+1}^n] - u_i^{(n)} \right\} \quad (10)$$

$$u_i^{(n+1)} = (1 - \omega)u_i^{(n)} + \omega \left\{ \frac{c_i}{(1+\lambda)} + \frac{\lambda}{2(1+\lambda)} [u_{i-1}^{(n+1)} + u_{i+1}^n] \right\}$$

Which is called successive over relaxation method. ω is called the relaxation factor or accelerating factor and it lies , generally, between 1 & 2 [14].

Let us consider a boundary value problem

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial X^2} = 0 \quad (11)$$

$$\text{With } U(X, 0) = 0, U(0, t) = 1, U(1, t) = 0, 0 \leq X \leq 1$$

Using Successive over Relaxation method for (11), we have

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$$U_{m,n+1} = (1 - \omega)U_{m,n} + \omega \left[\frac{r}{2(1+r)} (U_{m+1,n} + U_{m-1,n+1}) + \frac{\lambda_i}{(1+r)} \right] \quad (12)$$

Where $r = \frac{k}{h^2}$ and $\lambda_i = U_{m,n} + \frac{r}{2} (U_{m+1,n} - 2U_{m,n} + U_{m-1,n})$

Taking $k = 0.01$, $h = 0.1$, $r = 1$

III. MATHEMATICAL SOLUTION WITH DIFFERENT VALUE OF ω

Numerical calculations for different values of U at different time t and different length X are shown in the following tables.

$\omega = 1$

t→ X↓	t=0.01	t=0.02	t=0.03	t=0.04	t=0.05
	U				
0	1	1	1	1	1
0.1	0.25	0.53125	0.601563	0.658691	0.692749
0.2	0.0625	0.203125	0.317383	0.385498	0.439606
0.3	0.015625	0.068359	0.140869	0.203491	0.254372
0.4	0.0039063	0.021484	0.055542	0.096191	0.135167
0.5	0.0009766	0.00647	0.020203	0.041403	0.066111
0.6	0.0002441	0.001892	0.006939	0.016543	0.029996
0.7	6.104E-05	0.000542	0.002284	0.006234	0.012756
0.8	1.526E-05	0.000153	0.000728	0.002243	0.005136
0.9	3.815E-06	4.24E-05	0.000226	0.000776	0.00197
1	9.537E-07	1.16E-05	6.71E-05	0.00025	0.000686

$\omega = 1.27$

t→ X↓	t=0.01	t=0.02	t=0.03	t=0.04	t=0.05
	U				
0	1	1	1	1	1
0.1	0.3175	0.613287	0.652693	0.713499	0.736403
0.2	0.1008063	0.288631	0.401144	0.463067	0.515135
0.3	0.032006	0.121458	0.216709	0.28318	0.336192
0.4	0.0101619	0.04803	0.105992	0.160742	0.20716
0.5	0.0032264	0.018255	0.048259	0.084897	0.120649
0.6	0.0010244	0.00675	0.020849	0.042145	0.066503
0.7	0.0003252	0.002446	0.008657	0.019876	0.034836
0.8	0.0001033	0.000873	0.003485	0.008987	0.017428
0.9	3.279E-05	0.000308	0.001367	0.00391	0.008308
1	1.041E-05	0.000105	0.000503	0.00154	0.003464

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$$\omega = 1.47$$

t→ X↓	t=0.01	t=0.02	t=0.03	t=0.04	t=0.05
	U				
0	1	1	1	1	1
0.1	0.3675	0.661541	0.68219	0.747189	0.760679
0.2	0.1350563	0.351177	0.452814	0.512731	0.561297
0.3	0.0496332	0.16877	0.272458	0.337605	0.391039
0.4	0.0182402	0.076617	0.150779	0.210895	0.260242
0.5	0.0067033	0.03352	0.078316	0.124607	0.16599
0.6	0.0024635	0.01429	0.038776	0.069949	0.101527
0.7	0.0009053	0.005976	0.018503	0.037563	0.059637
0.8	0.0003327	0.002462	0.008576	0.019421	0.033686
0.9	0.0001223	0.001003	0.003874	0.00965	0.018079
1	4.493E-05	0.000392	0.001608	0.004214	0.00821

$$\omega = 1.67$$

t→ X↓	t=0.01	t=0.02	t=0.03	t=0.04	t=0.05
	U				
0	1	1	1	1	1
0.1	0.4175	0.700821	0.708534	0.77577	0.78125
0.2	0.1743063	0.410879	0.49759	0.558103	0.600837
0.3	0.0727729	0.220927	0.325499	0.388875	0.441391
0.4	0.0303827	0.112855	0.19905	0.261173	0.312221
0.5	0.0126848	0.055725	0.115312	0.168348	0.213638
0.6	0.0052959	0.026859	0.064029	0.104238	0.141596
0.7	0.002211	0.012714	0.034383	0.062223	0.090945
0.8	0.0009231	0.005935	0.017974	0.035927	0.056451
0.9	0.0003854	0.002739	0.009146	0.01985	0.033049
1	0.0001609	0.001197	0.00416	0.009318	0.015842

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$$\omega = 1.87$$

t→ X↓	t=0.01	t=0.02	t=0.03	t=0.04	t=0.05
	U				
0	1	1	1	1	1
0.1	0.4675	0.732625	0.733745	0.799472	0.800305
0.2	0.2185563	0.466448	0.537787	0.599836	0.635468
0.3	0.102175	0.276009	0.375327	0.437892	0.487555
0.4	0.0477668	0.156124	0.248757	0.311248	0.362675
0.5	0.022331	0.085652	0.15774	0.214775	0.26265
0.6	0.0104397	0.045963	0.096477	0.143792	0.185556
0.7	0.0048806	0.024255	0.0573	0.093539	0.12783
0.8	0.0022817	0.012633	0.033224	0.059078	0.085174
0.9	0.0010667	0.006511	0.01868	0.035382	0.052907
1	0.0004987	0.003109	0.009072	0.017381	0.026153

III. CONCLUSION

For $\omega = 1$, it works as Gauss Seidal method. It was shown by B.A.Carre that for $\omega = 1.875$ the rate of convergence of (12) is twice as fast as when $\omega = 1$. In general, however it is difficult to estimate the best value of ω .

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